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Exploring a Mathematics Teacher's Conceptions of Infinity: The Case of Louise

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Abstract

Several papers studied infinity from the difficulties that students and teachers show in developing the concept. For this study, it was considered the analysis of the equality $0.999\ldots = 1$. Mainly, this research aims to show that a mathematics teacher presents erroneous conceptions just like a student; that is, both students and teachers have the same difficulties in the concept of infinity. To this aim, a semi-structured interview was conducted with an in-service mathematics teacher in Tlaxcala, Mexico. The purpose of this research is to exhibit a high school math teacher's misconceptions about the concept of infinity. In general, misconceptions found here can be divided into four groups: without a clear picture of the concept of infinity, an infinite periodic decimal number cannot be a representation of a finite number, a decreasing infinite sum cannot lead to a finite number and an infinite process is limited in real life is finite and has ended. The results obtained were compared with those already available in references about the difficulties with students and teachers, finding that the results shown here are like those reported in the literature. This highlights the need to overcome the teacher's conceptions of infinity in future research.

Keywords: infinity, mathematics teachers, misconceptions

INTRODUCTION

The conception mathematics teachers have about infinity has been studied due to its presence in various of topics within the high school curriculum of the Mexican Secretariat of Public Education. For example, in sequences and series in algebra, slope in trigonometry, functions, limits, and integrals in differential and integral calculus.

The literature refers to studies that raise the most common difficulties in understanding infinity among students and in-service or prospective teachers. To name a few examples: its abstract nature (Manfreda Kolar and Čadež, 2012), its double dichotomy both actual and potential (Dubinsky et al., 2005a, 2005b; Monaghan, 2001), the lack of a pictorial or mental image of what is being represented (Ángeles-Navarro and Pérez-Carreras, 2010), large finite numbers deemed as infinites (Manfreda Kolar and Čadež, 2012; Medina Ibarra et al., 2019; Singer and Voica, 2003), or contradictory intuitions when working with infinity (Fuentes and Oktac, 2014; Tirosh, 2002; Wijeratne and Zazkis, 2015).

However, most studies do not show a possible relationship between teachers' and students' conceptions. Thus, this research aims to reveal that teachers also have similar conceptions as their students about infinity, and that subsequently, these are transmitted by teachers to their students as warn Manfreda Kolar and Čadež (2012). Similarly, Schwarzenberger and Tall (1978) theorized that if teachers held misconceptions and communicated this unease to students, this could be one source of students' difficulties about the infinity.

Therefore, this work raises the following research question: What misconceptions might a high school math teacher have regarding the concept of infinity? For this study, a semi-structured interview was conducted to examine an in-service high school mathematics teacher's conception of infinity. The point is undoubtedly not to generalize the results, but to contrast them with those reported in the literature, especially those associated with the equality 0.999 … = 1 (Ángeles-Navarro and Pérez-Carreras, 2010; Eisenmann, 2008; Hannula et al., 2006; Mena-Lorca et al., 2015; Schwarzenberger and Tall, 1978; Yopp et al., 2011).

LITERATURE REVIEW

According to Manfreda Kolar and Čadež (2012) "potential infinity is related to an ongoing process without an end", and, on the other hand, "actual infinity attributes a finite entity to this infinite process" (p. 390).

There are several studies that approach infinity from different perspectives; however, all of these perspectives focused on the difficulties that students and teachers show in developing the concept. For this study, it was considered the analysis of the equality 0.999 … = 1 (Ángeles-Navarro and Pérez-Carreras, 2010; Eisenmann, 2008; Hannula et al., 2006; Mena-Lorca et al., 2015; Schwarzenberger and Tall, 1978; Yopp et al., 2011), given that similar studies argue that, despite the evidence, some teachers refuse to accept the completeness of the process of infinite decimals method or, when rounding amounts due to the impossibility of completeness justify that the small amounts are irrelevant (Ángeles-Navarro and Pérez-Carreras, 2010; Kattou et al., 2010; Yopp et al., 2011).

For example, in a study conducted with in-service teachers, it was found that about 72% of teachers have an image of infinity as an endless process (potential infinity), and only 28% define infinity as an object (actual infinity) (Kattou et al., 2010, p. 1775). On the other hand, as the work of Manfreda Kolar & Čadež (2012) shows, there is a tendency to interpret the problem with potential infinity and not with actual infinity in teachers who simultaneously transmit this knowledge to their students. Therefore, it is important to identify the conceptions present in a teacher in service of mathematics.

METHOD

This research presents a single case study: A semi-structured interview exploring possible misconceptions that the mathematics teacher might have. The interviewee was asked for permission to record the audio of the interview for later transcription and analysis. For further analysis of information researchers followed an open data exploration and typological analysis (Hatch, 2002) concentrating primarily on the formulation of the obstacles in the understanding of infinity and the process of their overcoming. Excerpts of the interview are shown in the results and discussion.

Instrument

The interview lasted for 30 minutes and began with the following triggering situation: "*Imagine you had a box of sticks; the first stick is one meter long, and each following stick is half as long as the one before. If you had to connect every single stick by their extreme ends one after the other, how long would the combination be?*" This question was taken and modified from Belmonte and Sierra (2011, p. 161). This type of question suggests the presence of potential infinity, which helped in demonstrating the conceptions of the teacher because, according to Manfreda Kolar and Čadež (2012), "there is no infinity in our surroundings —the only way to imagine it in real life is to see it as a result of infinite processes or an indefinite repetition of a certain process" (p. 410).

Participant

For the choice of the informant, a professor in service of high school level who taught at least differential or integral calculus was considered since infinity is expected to appear when working with limits, the definition of derivative or definite integral. In addition, a professor considered an 'expert' because of his high academic degree would suggest that he would have a more solid conception of infinity. The informant (referred to by the pseudonym 'Louise' hereafter) taking part in this case study was a 38-year-old teacher with a PhD and a Master of Chemical Engineering since 2011. She has taught chemistry, physics, and mathematics in three different high school institutions in the state of Tlaxcala, Mexico over three consecutive years, from 2017 to 2019. She currently works as a mathematics teacher in a private high school institution, teaching algebra and integral calculus.

RESULTS

The interview made clear that the interviewee experienced the cognitive conflict of an infinite process that could be considered concluded and assimilated. This matches with the results from the research of Manfreda Kolar and Čadež (2012) where the interviewees had strong problems solving tasks of the type 'infinitely close'.

In addition, what was reported by students regarding the expression 0.999 … = 1 (Ángeles-Navarro and Pérez-Carreras, 2010; Eisenmann, 2008; Hannula et al., 2006; Mena-Lorca et al., 2015; Schwarzenberger and Tall, 1978; Yopp et al., 2011) agrees with what was explored in this case study. The teacher indicates that both numerical representations are not the same, however, by suggesting an activity involving 1/3 = 0.333 … she manages to accept the equivalence of 0.999 … and 1.

In response to the research question after the analysis of the interview, four misconceptions are catalogued: without a clear picture of the concept of infinity, an infinite periodic decimal number cannot be a representation of a finite number, a decreasing infinite sum cannot lead to a finite number and an infinite process is limited in real life and has ended. More details regarding this research's particular results are shown below.

DISCUSSION

Without a clear picture of the concept of infinity

The first answer obtained from Louise when posing the initial question was:

"When should I stop measuring the half of each stick?".

According to Belmonte and Sierra (2011), this answer could possibly convey an incomplete conception of infinity: "We do not know how much is infinity; in other words, we do not know how many numbers are to be added" (p. 161). That is, it presents a limitation in the conception of infinity at first without even conceiving it as an endless process, as potential infinity. Afterwards, when mentioning to the teacher that it is an indefinitely continuous process, she responds by saying it continues until infinity.

An infinite process is limited in real life and has end

When posing her the question, at another moment in the interview, if she considers there is a finite or infinite number of sticks, Louise then mentions the sticks must be finite:

"*Because there will be a moment where the stick cannot be further… further split… according to the measurement*".

This matches with that reported by Manfreda Kolar and Čadež (2012) "If students attempt to apply the descriptions to real-life situations, they face the following problem: The abstract mathematical description which represents an infinite set becomes finite when transferred to a concrete physical situation" (p. 402), and, as mentioned in a different segment "The limitations of the physical world are one of the main reasons for difficulties related to understanding of geometrical problems on infinity" (p. 398).

A decreasing infinite sum cannot lead to a finite number

The previous excerpts contradict what the teacher mentions at the beginning as an observation of an infinite process of the sum of sticks. In order to observe this fact and make the teacher aware of it, she was asked to indicate which would be the last stick in the box if she considers it finite. During the questioning, it was evident she was incapable of finding an answer after trying to make several calculations and even suggesting an expression that would determine the whole sequence:

'*I would express it as… [three seconds]* $\frac{1}{2^n}$ *[mentions one divided by two to the power of* n*]*'.

Nonetheless, Louise still was could not visualize a last term using this expression, and it was not enough to change her idea that there is not a last stick (it does not exist as a last stick). Although previously the teacher indicates that there is a last rod, she is later unable to find it, even so she does not consider the option of changing her mind. Likewise, "intuitions of actual infinity are very resistant to the effects of age and school-based instruction" (Tirosh, 2002, p. 201).

Another important observation is that Louise reflects a conception of potential infinity when having enough time to add "every single" fraction. This fact influences its answer to the posed question:

'*It should be less than two*'.

Louise remarks that this happens because, when adding the fractions $\frac{1}{2}+\frac{1}{4}=\frac{3}{4}$, and when adding progressively smaller fractions, it should not be larger than the unit.

From this point onwards, the teacher is posed the question about whether the result of the addition $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$ would have a value approaching the unit or precisely the same as the unit in different occasions. In multiple occasions, Louise answered that it certainly was approaching the unit; for example:

'*Well… I would say it is approaching if we are representing progressively smaller portions*'.

This reinforces the results shown in Ángeles-Navarro and Pérez-Carreras (2010): "students called 'potentialist' see 0.999 … as an infinite process that is being constructed in time approaching 1 but never reaching it" (p. 32).

At some other point, Louise justifies her approximation by highlighting that there is a mistake in every method, which in turn tells us that "the students think that the difference between 0.999 … and 1 is infinitesimally small… or the number 0.999 … is the last number before 1" (Eisenmann, 2008, p. 35).

An infinite periodic decimal number cannot be a representation of a finite number

Lastly, the expression 0.999 … = 1 was shown to the teacher with the goal of triggering a change in her conception of infinity through means of two different representations of the same number. This was widely reported in the literature (Ángeles-Navarro and Pérez-Carreras, 2010; Eisenmann, 2008; Hannula et al., 2006; Mena-Lorca et al., 2015; Schwarzenberger and Tall, 1978; Yopp et al., 2011).

Just like the previously mentioned studies, similar results were obtained. The teacher did not accept the idea of both representations being the same by mentioning that the left one is an approximation; however, when showing her the expressions 1/3 = 0.333 … o 2/3 = 0.666 …, she accepted them with ease. This is consistent with the results shown in Edwards (1997), which indicate that the majority of students tend to reject the first equality rather than the second. Moreover, students seem to accept that 0.333 … tends to 1/3 as a result of dividing 1 by 3, something not feasible in the case of the expression 0.999 … = 1.

To achieve the expression 0.999 … = 1 being accepted, the teacher was asked to add the expressions corresponding to a third of two thirds in its rational form and, afterwards, in its infinite periodic decimal form with the goal of making her realize that the equality 0.999 … = 1 is correct. Louise responded, astonished, that it is equal to one and reflecting on the original problem of the sticks, said that her approximation of "almost one" is, in fact, precisely the unit.

In this way, at the end a change in the concept of infinity was shown, with an answer explaining that the total length of the sticks would be of two meters; however, just as indicated by Singer and Voica (2003), even after the two interventions that happened in their research, the intuitions about infinity in the interviewees was persistent. Thus, the notion of potential infinity should not be considered as something left behind, but rather as a tool to identify the ideas to be corrected in future research.

CONCLUSIONS

As mentioned at the beginning, the main objective was showing inaccurate ideas surrounding infinity in a high school mathematics teacher, and that, at the same time, are underpinned in the literature. In conclusion, we can say that the main inaccurate ideas are: 1) Not having a clear image of the concept of infinity; in other words, not knowing how far to stop estimating; 2) Considering that an infinite periodic decimal cannot be a representation of a finite number; in other words, an approximation, truncation, or rounding with certain margin for error; 3) Considering that a decreasing finite addition cannot result in a finite number; 4) Believing an infinite process to be limited to having an end in real life.

At the end of the interview, the teacher tended to accept the equality 0.999 … = 1 and that considering all the rods as a complete process, the total length would be 2 meters. However, it is not enough to develop a single situation to alleviate the conception that it presents about the infinite.

On this basis, future research could design tasks that helps the teacher on overcoming the misconceptions of infinity because, just as some authors indicate, this type of conceptions are persistent and are transmitted to the students, which they in turn can communicate further in their environment and create a cycle of reinforcement. (Kattou et al., 2010; Manfreda Kolar & Čadež, 2012; Mena-Lorca et al., 2015; Schwarzenberger & Tall, 1978; Yopp et al., 2011).

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