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How to Construct Super Edge-Magic Total Labeling of Theta Graph $\theta(2, b, c)$

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Abstract

This research study and provide the property of super edge-magic total labelings of theta graph. Edge magic labeling on a graph G is an injective function λ from $V(G) \cup E(G)$ to a subset of integers $\{1,2,\ldots,|V(G) \cup E(G)|\}$ with the property that there is a positive integer k such as $\lambda(x) + \lambda(y) + \lambda(xy) = k$ for each $xy \in E(G)$. An edge-magic labeling λ is called super edge-magic total labeling if it satisfies $\lambda(V(G)) = \{1,2,\ldots,|V(G)|\}$. A graph is called (super) edge-magic total if it admits some (super) edge-magic total labeling. A theta graph $\theta(a, b, c)$ is constructed by embedding the endpoints of three paths of length consecutive $a, b, and\ c$ such that there are two vertices of degree three and the other of degree two. This study gave some conditions for such a super edge-magic total of theta graph. Based on this condition, this paper introduce some algorithms to apply and develop super edge-magic total labeling from some previous theta graphs.

Keywords: Edge magic labeling, super edge-magic total labeling, theta graph

INTRODUCTION

A general reference for graph-theoretic ideas can be seen in Hartsfield & Ringel (1994). In 2001 Wallis introduced a magic graph and some magic graph labeling algorithms (Slamin et al., 2002). An edge magic labeling on graph G is an injective function λ from $V(G) \cup E(G)$ to a subset of integers $\{1,2,\ldots,|V(G) \cup E(G)|\}$ with the property that there is a positive integer k such as $\lambda(x) + \lambda(y) + \lambda(xy) = k$ for each $xy \in E(G)$ (Figueroa-Centeno, Ichishima, & Muntaner-Batle, 2001; Li et al., 2020). An edge-magic labeling λ is called super edge-magic total labeling if it satisfies $\lambda(V(G)) = \{1,2,\ldots,|V(G)|\}$. A graph is called (super) edge-magic total if it admits any (super) edge-magic total labeling (Takahashi, Muntaner-Batle, & Ichishima, 2023).

A theta graph $\theta(a, b, c)$ is a graph constructed by embedding the endpoints of three paths of length consecutive $a, b, and\ c$ such that there are two vertices of degree three and the other vertices of degree two (Swaminathan & Jeyanthi, 2006).

Let $V(\theta(a, b, c))$ and $E(\theta(a, b, c))$ be the vertex and the edge set of $\theta(a, b, c)$ for positve integers $a, b, c,\ a \le b \le c$.
We define:
$V(\theta(a,b,c)) = \{x_h, y_i, z_j \mid h \in [0, a],\ i \in [1, b-1],\ j \in [1, c-1]\}$; and
$E(\theta(a,b,c)) = \{x_h x_{h+1},\ x_0 y_1,\ y_i y_{i+1}, x_0 z_1, z_j z_{j+1}, y_{\beta-1} x_h,\ z_{\gamma-1} x_h, for\!: h \in [0, a-1],\ i \in [1, b-2], j \in [1, c-2]\}$

Using the above set, can be obtain x_0 and x_a as the vertices of degree three. Figure 1 shows the theta graph $\theta(3,4,5)$.

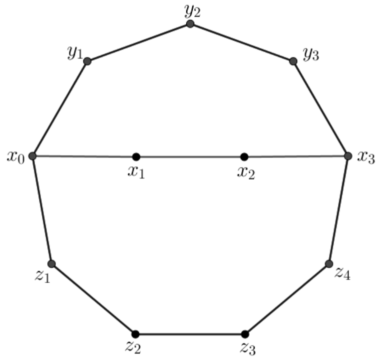

Figure 1. Theta Graph $\theta(3,4,5)$

The super edge-magic total labelings of a theta graph were studied by Swaminathan and Jeyanthi (2006), but the results obtained are still limited to a few graph theta. It can be checked in A Dynamic Survey of Graph Labeling (Gallian, 2018). Some labeling techniques of theta graphs have been studied. Putra and Susanti, 2018 studied the total edge irregularity strength of uniform theta graphs. Sugumaran (2017) provided sum divisor cordial labeling of theta graph. In this research, we focus on develop a similar notion for edge-magic total labelings on a given theta graph.

Based on these conditions, it can be introduce some algorithms for super edge-magic total labeling of theta graphs. By using these algorithms, we can provide more evidence to construct the super edge-magic total labeling of theta graphs $\theta(2,b,c)$, especially theta graphs $\theta(2,2,c)$ and some other theta graphs.

METHOD

This study uses a literature study research method and focuses on graph theory literature. Based on the literature that has been studied, we will find and formulate a graph labeling technique on theta graphs $\theta(2,b,c)$, especially on theta $\theta(2,2,c)$, another theta graph.

RESULTS AND DISCUSSION

Magic Number of Theta Graph

Let G be a simple and finite graph. An edge magic labeling on graph G is an injective function λ from $V(G) \cup E(G)$ to a subset of integers $\{1,2,\dots,|V(G) \cup E(G)|\}$ with the property that there is a positive integer k such as $\lambda(x)+\lambda(y)+\lambda(xy)=k$ for each $xy \in E(G)$ (Figueroa-Centeno et al., 2001; Li et al., 2020).

An edge-magic labeling λ is called super edge-magic total labeling if it satisfies $\lambda(V(G))=\{1,2,\dots,|V(G)|\}$ with $|V(G)|=v$. A super edge-magic total labeling of G admits any (super) edge-magic total labeling (Takahashi et al., 2023).

Swaminathan and Jeyanthi (2006) showed if theta graph is a super edge-magic graph with v vertices, so it is obtained:

$$\lambda(x_0)+\lambda(x_a)=v+1 \text{ ; } \textit{if } v \textit{ is even}, \text{ and}$$
$$\lambda(x_0)+\lambda(x_a)=\frac{v+1}{2}; \text{ or } \lambda(x_0)+\lambda(x_a)=\frac{3(v+1)}{2}; \text{ if } v \text{ is odd}$$

where x_0 and x_a are the vertices of degree three

then we construct theta graph $\theta(a,b,c)$ magic number as in the following theorem:

Theorem 1. *Let theta graph $\theta(a,b,c)$ with v vertices be a super edge-magic total graph with labeling λ and magic number k. Then, the labeling λ defined:*

$$k=\begin{cases}\frac{5v+4}{2}; \; if\ v\ is\ even, an \\ \frac{5v+3}{2}\ or\ \frac{5v+5}{2}\ ; \; if\ v\ is\ odd\end{cases}$$

Proof. Let x_0 and x_a be the vertices of degree three in $\theta(a,b,c)$. We can construct:

$$k=\frac{5v+2}{2}+\frac{\lambda(x_0)+\lambda(x_a)}{(v+1)} \text{ .... (1)}$$

where$\lambda(x_0)$ and $\lambda(x_a)$ are the labels of degree three vertices. Then, we will show the magic number k for any vertex v.

Case (i). v is even

It is easy to see that the magic number $k=\frac{5v+2}{2}$ is a positive integer. We will also check $\frac{\lambda(x_0)+\lambda(x_a)}{(v+1)}$ is a positive integer.

As $\theta(a,b,c)$ has v vertices, we obtain $\lambda(x_0)+\lambda(x_a)\leq 2v-1$. Let $\frac{\lambda(x_0)+\lambda(x_a)}{(v+1)}=m$ for any integer m. This mean $\lambda(x_0)+\lambda(x_a)=m(v+1)$. Since $\lambda(x_0)+\lambda(x_a)\leq 2v-1$, then $m=1$, and we found the fact $\lambda(x_0)+\lambda(x_a)=v+1$. Therefore if $\lambda(x_0)+\lambda(x_a)=v+1$ is substituted to equation (1), we obtain $k=\frac{5v+4}{2}$.

Case (ii). v is odd

Let $v = 2n + 1$ for any integer n. By substituting to equation (1),we obtain

$$k = \frac{10n + 7}{2} + \frac{\lambda(x_0) + \lambda(x_a)}{2(n + 1)};$$
$$\Leftrightarrow \quad k = \frac{8n+4}{2} + \frac{2n}{2} + \frac{2}{2} + \frac{1}{2} + \frac{\lambda(x_0)+\lambda(x_a)}{2(n+1)};$$
$$\Leftrightarrow \quad k = 2v + n + 1 + \frac{1}{2}\left(1 + \frac{\lambda(x_0)+\lambda(x_a)}{(n+1)}\right);$$

It is easy to see that $2v + n + 1$ is an integer. Then, we will show $\frac{1}{2}\left(1 + \frac{\lambda(x_0)+\lambda(x_a)}{(n+1)}\right)$ must be an integer. When $\frac{1}{2}\left(1 + \frac{\lambda(x_0)+\lambda(x_a)}{(n+1)}\right)$ is any integer, shall be $1 + \frac{\lambda(x_0)+\lambda(x_a)}{(n+1)}$ is an even integer hence $\frac{\lambda(x_0)+\lambda(x_a)}{(n+1)}$ is an odd integer.

Let $\frac{\lambda(x_0)+\lambda(x_a)}{(n+1)} = q$ for any odd integer q, then $\lambda(x_0) + \lambda(x_a) = q(n + 1)$. Since $v = 2n + 1$, then $\lambda(x_0) + \lambda(x_a) = q\left(\frac{v+1}{2}\right)$, We obtain fact that $\lambda(x_0) + \lambda(x_a) \le 2v - 1$, therefore $q = 1$ or $q = 3$ and $\lambda(x_0) + \lambda(x_a) = \frac{v+1}{2}$ or $\lambda(x_0) + \lambda(x_a) = \frac{3(v+1)}{2}$. By substituting it to equation (1), we obtain $k = \frac{5v+3}{2}$ *or* $k = \frac{5v+5}{2}$.

Consider the following cases showen, we obtain $\boldsymbol{k} = \frac{\mathbf{5v+4}}{\mathbf{2}}$*; if* $\boldsymbol{v}$ *is even, and* $\boldsymbol{k} = \frac{\mathbf{5v+3}}{\mathbf{2}}$*; or* $\boldsymbol{k} = \frac{\mathbf{5v+5}}{\mathbf{2}}$ *; if* $\boldsymbol{v}$ *is odd* ∎

Necessary Conditions

We know that theta graph $\theta(a, b, c)$ isconstructed by embedding the end points of three paths of length consecutive $a, b, and\ c$. We also defined x_0 and x_a as the vertices of degree three.

Further, in order to know what possible conditions to construct super edge-magic total labeling of theta graphs, we add the following necessary conditions as in the following lemma:

Lemma 1. *Let $a, b,$ and c be the length of three paths that construct theta graph $\theta(a, b, c)$. Theta graph is a super edge-magic total graph if there are no two or more paths with a lenght of one.*

Proof: Since $\theta(a, b, c)$ is the super edge-magic total graph with injective function λ from $V(\theta(a, b, c)) \cup E(\theta(a, b, c))$ to a subset of integers $\{1,2, . . , |V(\theta) \cup E(\theta)|\}$ (we notice $V(\theta(a, b, c) = V(\theta)$ and $E(\theta(a, b, c) = E(\theta))$.

So we receive the label $\lambda(V(\theta)) = \{1,2, \ldots, |V(\theta)|\}$. Because of this condition, the edge labels must be the consecutive integer $\{(|V| + 1), \ldots, |V \cup E|\}$. If there are two paths with length one, then it is impossible to label the two paths with different edge labels because the two edges are at the same two points. So, it is impossible that there are two or more paths with a length of one to construct this labeling.

∎

Lemma 2. *Let b and c be the positive integers with $2 \le b \le c$. If theta graph $\theta(2, b, c)$ is the super edge-magic total graph, then $b + c \ne 4$ or $b + c \ne 6$.*

Proof. Let $\theta(2, b, c)$ be the super edge-magic total labeling λ with magic number k. We construct x_0 and x_a as the vertices of degree three of $\theta(2, b, c)$,

Case (i). $b + c \ne 4$

Let $b + c = 4$, consider the theorem about the sum of degree three vertices in [3], then we obtain $(\lambda(x_0), \lambda(x_a)) \in \{(1,2), (4,5)\}$ where x_0 and x_a are the degree three vertices of $\theta(2, b, c)$.

Since $(\lambda(x_0), \lambda(x_a)) \in \{(1,2)\}$, then we obtain $k = 14$ and $\lambda(x) + \lambda(y) \in [3,8]$ for each $xy \in E(\theta(a, b, c))$.

The condition of $\lambda(x) + \lambda(y) = 3$ occurred only when $(\lambda(x), \lambda(y)) \in \{(1,2)\}$, however, $(\lambda(x_0), \lambda(x_a)) \in \{(1,2)\}$, thus there is no super edge-magic total labeling.

Case (ii). $b + c \neq 6$

Let $b + c = 6$, consider the theorem about the sum of degree three vertices in [3], then we obtain $\left(\lambda(x_0), \lambda(x_a)\right) \in \{(1,3), (5,7)\}$. Since $\left(\lambda(x_0), \lambda(x_a)\right) \in \{(1,3)\}$, then we obtain $k = 19$ and $\lambda(x) + \lambda(y) = [4,11]$for each $xy \in E(\theta(a, b, c))$.

The condition of $\lambda(x) + \lambda(y) = 4$ occurred only when $(\lambda(x), \lambda(y)) \in \{(1,3)\}$, however $(\lambda(x_0), \lambda(x_a)) \in \{(1,3)\}$.

When $(\lambda(x_0), \lambda(x_a)) \in \{(5,7)\}$, thus $k = 20$ and $\lambda(x) + \lambda(y) \in [5,12]$. The condition of $\lambda(x) + \lambda(y) = 12$ occurred only when $(\lambda(x), \lambda(y)) \in \{(5,7)\}$; however $(\lambda(x_0), \lambda(x_a)) \in \{(5,7)\}$, thus there is no super edge-magic total labeling. ■

In the next discussion, there will be constructed the super edge-magic total labeling of $\boldsymbol{\theta}(\mathbf{2}, \mathbf{2}, \boldsymbol{c})$ for any positive integer $\boldsymbol{c}$ by using the necessary and sufficient conditions.

Constructing Super Edge-Magic Total Labeling of $\boldsymbol{\theta}(\mathbf{2}, \mathbf{2}, \boldsymbol{c})$

In this discussion, we give algorithms to construct super edge-magic total labeling of each positive integer of c.

We have defined:

$$V\left(\theta(a, b, c)\right) = \left\{x_h, y_i, z_j \mid h \in [0, a],\ i \in [1, b - 1],\ j \in [1, c - 1]\right\} \text{; and}$$

$$E\left(\theta(a, b, c)\right) = \{x_h x_{h+1},\ x_0 y_1,\ y_i y_{i+1},\ x_0 z_1,\ z_j z_{j+1},\ y_{\beta-1} x_h,\ z_{\gamma-1} x_h\}$$

for:$h \in [0, a - 1]$, $i \in [1, b - 2]$, $j \in [1, c - 2]\}$.

where x_0 and x_a are the vertices of degree three

Theorem 2. *For $c \not\equiv 2 \ (mod\ 4)$, theta graph $\theta(2,2, c)$ is a super edge-magic total graph.*

Proof. Let c be the positive integer where $2 \leq c$, we will show $\theta(2,2, c)$ with v vertices and e edges is the super edge-magic total graph.

To construct the super edge-magic total labeling, We first have to find the magic number k of this labeling. We divided it into two cases.

Case (i). *c is odd*

By using Theorem 1, we obtain $k = \frac{5v+3}{2}$, we define super edge-magic total $\lambda_1 \colon V\left(\theta(2,2, c)\right) \rightarrow \{1,2, \dots, v\}$, thus

$$\lambda_1(x_h) = \begin{cases} \dfrac{c+3}{2}, & for\ h = 0; \\ c + 3, & for\ h = 1; \\ \dfrac{c+5}{2}, & for\ h = 2. \end{cases}$$

$$\lambda_1(y_i) = 1 \quad for\ i = 1$$

$$\lambda_1(z_j) = \begin{cases} \frac{2c+5-j}{2} & for\ 1 \leq j \leq c - 2\ \ and\ j\ is\ odd; \\ \frac{c+3-j}{2}, & for\ 2 \leq j \leq c - 1\ \ and\ j\ is\ even. \end{cases}$$

By using those algorithm, the label of vertices will establish an arithmetic sequence $(\{1,2,3, \dots, v\})$. So, we can give the label to the edge, which is the sum of the neighbours' vertices. The label is an arithmetic sequence with a difference of one. Therefore the labels of the edges is $\{(v + 1), (v + 2), \dots, (v + e)\}$ and make sure their sum labels are equal to.

Case (ii). *c is even*, $c \equiv 0\ (mod\ 4)$

From Lemma 2, we know $c \neq 2$ and $c \neq 4$. By using Theorem 1, we obtain $k = \frac{5v+4}{2}$

In this following condition, we define super edge-magic total $\lambda_2 \colon V\left(\theta(2,2, c)\right) \rightarrow \{1,2, \dots, v\}$, thus:

$$\lambda_2(x_h) = \begin{cases} \frac{c}{4} + 2, & for\ h = 0; \\ \frac{3c}{4} + 3, & for\ h = 1; \\ \frac{c}{4}, & for\ h = a = 2 \end{cases}$$

$$\lambda_2(y_1) = \frac{3c}{4} + 2$$

$$\lambda_2(z_j) = \begin{cases} \frac{c}{4}+j, & for\ j=1; \\ \frac{3c}{4}+5+\frac{j-2}{2}, & \begin{cases} for\ j=2,6,10,\dots,\frac{c}{2}-2\ and\ c\equiv 0\ (mod\ 8); \\ for\ j=2,6,10,\dots,\frac{c}{2}-4\ and\ c\equiv 4\ (mod\ 8); \end{cases} \\ \frac{c}{4}+2+\frac{j+1}{2}, & \begin{cases} for\ j=3,7,\dots,\frac{c}{2}-1\ and\ c\equiv 0\ (mod\ 8); \\ for\ j=5,9,\dots,\frac{c}{2}-1\ and\ c\equiv 4\ (mod\ 8); \end{cases} \\ \frac{c}{4}+j, & for\ j=3\ and\ c\equiv 4\ (mod\ 8); \\ \frac{3c}{4}+3+\frac{j-2}{2}, & \begin{cases} for\ j=4,8,\dots,\frac{c}{2}\ and\ c\equiv 0\ (mod\ 8); \\ for\ j=4,8,\dots,\frac{c}{2}-2\ and\ c\equiv 4\ (mod\ 8); \end{cases} \\ \frac{c}{4}+\frac{j+1}{2}, & \begin{cases} for\ j=5,9,\dots,\frac{c}{2}+1\ and\ c\equiv 0\ (mod\ 8); \\ for\ j=7,11,\dots,\frac{c}{2}+1\ and\ c\equiv 4\ (mod\ 8); \end{cases} \\ \frac{j}{2}-\frac{c}{4}, & for\ \frac{c}{2}+2\le j\le c-2\ and\ j\ is\ even; \\ \frac{j+3}{2}+\frac{c}{4}, & for\ \frac{c}{2}+3\ \le j\le c-1\ and\ j\ is\ odd. \end{cases}$$

Depending on these algorithms, the label of vertices will establish an arithmetic sequence ($\{1,2,3,\dots,v\}$). So we can give the label to the edge, whose sum of neighbours vertices label also established one difference arithmetic sequence. Therefore the labels of the edges is $\{(v+1),\ (v+2),\dots,\ (v+e)\}$ and make sure their sum labels is equal to *k*. ■

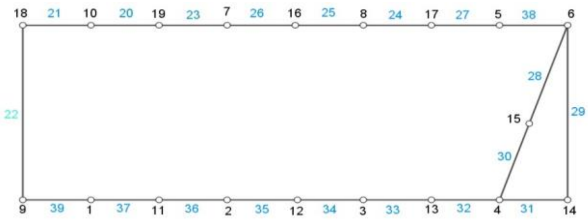

Figure 2. Super Edge-Magic Total Labeling of $\theta(2,2,16)$ Using Theorem 2

Constructing Super Edge-Magic Total Labeling of Another One

In this section, we give algorithms to construct new super edge-magic total graphs by extending the old ones.

Theorem 3. *Theta graph $\theta(2,b,c)$ be a super edge-magic total graph for $b+c\equiv 1\ (mod\ 4)$*

Proof. Let *b* and *c* be the positive integer where $b\le c$. Because of $b+c\equiv 1\ (mod\ 4)$, $\theta(2,b,c)$ has even v vertices. From theorem 1, we obtain $k=\frac{5v+4}{2}$. To construct this labeling, we will divide it into two cases.

Case (i). $b\equiv 0\ (mod\ 4),\ c\equiv 1\ (mod\ 4)$ or $b\equiv 1\ (mod\ 4),\ c\equiv 0\ (mod\ 4)$

The conditions of $b\equiv 0\ (mod\ 4),\ c\equiv 1\ (mod\ 4)$ and $b\equiv 1\ (mod\ 4),\ c\equiv 0\ (mod\ 4)$ are the same, so we choose $b\equiv 0\ (mod\ 4),\ c\equiv 1\ (mod\ 4)$

We define super edge-magic total $\lambda_3: V(\theta(2,b,c))\rightarrow\{1,2,\dots,v\}$, thus

$$\lambda_3(x_h) = \begin{cases} \frac{b+c-1}{2}-\frac{b-4}{4}, & for\ h=0; \\ b+c-\frac{b-4}{4}, & for\ h=1; \\ \frac{b+c+5}{2}+\frac{b-4}{4}, & for\ h=2. \end{cases}$$

$$\lambda_3(y_i) = \begin{cases} b+c+\frac{i+1}{2}-\frac{b-4}{4}, for\ 1\le i\le\frac{b}{2}-1, i\ is\ odd; \\ \frac{b+c+i-1}{2}-\frac{b-4}{4}, \quad for\ 2\le i\le\frac{b}{2}, i\ is\ even; \\ \frac{b+c+i}{2}-\frac{b-4}{4}, \quad for\ i=\frac{b}{2}+1; \\ \frac{i-1}{2}-\frac{b}{4}, \quad for\ \frac{b}{2}+2\le i\le b-1, i\ is\ odd; \\ \frac{b+c+i+1}{2}-\frac{b-4}{4}, \quad for\ \frac{b}{2}+1\le i\le b-2, i\ is\ even. \end{cases}$$

$$\lambda_3(z_j) = \begin{cases} b+c-\frac{j+1}{2}-\frac{b-4}{4}, for\ 1\le j\le\left\lfloor\frac{c}{2}\right\rfloor-3, j\ is\ odd; \\ \frac{b+c+1-j}{2}-\frac{b}{4}, \quad for\ 2\le j\le\left\lfloor\frac{c}{2}\right\rfloor-4, c>9, j\ is\ even \\ \frac{b+c-1-j}{2}-\frac{b}{4}, \begin{cases} for\ \left\lfloor\frac{c}{2}\right\rfloor-2\le j\le c-1, c>9\ j\ even; \\ for\ 2\le j\le c-1, c=9\ j\ is\ even; \end{cases} \\ \frac{b+c+2-j}{2}-\frac{b}{4}, \begin{cases} for\ j=\left\lfloor\frac{c}{2}\right\rfloor-1, c>9; \\ for\ j=3, c=9; \end{cases} \\ \frac{2b+2c+1-j}{2}-\frac{b-4}{4}, \ for\ \left\lfloor\frac{c}{2}\right\rfloor+1\le j\le c-2, j\ is\ odd. \end{cases}$$

By using these algorithms, the label of vertices will establish arithmetic sequence $1,2,3,\dots,v$. So we can give the label to the edge whosesum of neighbours vertices label established arithmetic sequence with a difference is also one. Further the labels of the edges is $\{(v+1),(v+2),\dots,(v+e)\}$ and certainly their sum labels is equal to *k*

Case (ii). $b\equiv 2\ (mod\ 4),\ c\equiv 3\ (mod\ 4)$ or $b\equiv 3\ (mod\ 4),\ c\equiv 2\ (mod\ 4)$

The conditions of $b\equiv 2\ (mod\ 4),\ c\equiv 3\ (mod\ 4)$and $b\equiv 3\ (mod\ 4),\ c\equiv 2\ (mod\ 4)$ are the same, so we choose $b\equiv 2\ (mod\ 4),\ c\equiv 3\ (mod\ 4)$

Same step as the previous theorem, we define super edge-magic total $\lambda_4 : V\big(\theta(2,b,c)\big) \to \{1,2,\dots,v\}$, thus,

$$\lambda_4(x_h) = \begin{cases} \frac{b+c+1}{2} - \frac{b-2}{4}, & for\ h = 0; \\ b+c+1 - \frac{b-2}{4}, & for\ h = 1; \\ \frac{b+c+3}{2} + \frac{b-2}{4}, & for\ h = 2. \end{cases}$$

$$\lambda_4(y_i) = \begin{cases} \frac{2(b+c)+i+1}{2} - \frac{b-6}{4}, for\ 1 \le i \le \frac{b}{2} - 2, b > 2\ and\ i\ is\ odd; \\ \frac{b+c+i-1}{2} - \frac{b-6}{4}, \quad for\ 2 \le i \le b-2\ and\ i\ is\ even; \\ \frac{i+1}{2} - \frac{b-2}{4}, \quad \begin{cases} for\ i = 1\ and\ b = 2; \\ for\ \frac{b}{2} \le i \le b-1, b > 2\ and\ i\ is\ even. \end{cases} \end{cases}$$

$$\lambda_4(z_j) = \begin{cases} \frac{2(b+c)+1-j}{2} - \frac{b-2}{4}, for\ 1 \le j \le c-2, j\ is\ odd; \\ \frac{b+c+1-j}{2} - \frac{b-2}{4}, \quad for\ 2 \le j \le c-1, j\ is\ even. \end{cases}$$

■

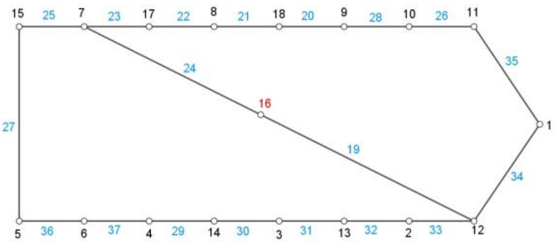

Figure 3. Super Edge-Magic Total Labeling of $\theta(2,8,9)$ Using Theorem 3

CONCLUSION

This research provided the labeling of total super magic edges on graph $\theta(2,b,c)$ for each natural number a, b and c with $a = 2$ and $2 \le b \le c$. Theorem has shown that $\theta(2,b,c)$ with v dots had a total labeling super-magic side with magic constant $k = \frac{5v+3}{2}$ or $\frac{5v+5}{2}$, for v odd and $k = \frac{5v+4}{2}$, for even v. This theorem also applies not limited to $a = 2$, but also applies to a natural number in general. Furthermore, Lemma 2 has been shown that if the theta graph $\theta(2,b,c)$ has super magic edge-total labeling, then $b + c \ne 4\ or\ b + c \ne 6$. The super magic edge-total labeling algorithm of theta graph $\theta(2,b,c)$, especially theta graph $\theta(2,2,c)$ and another theta graph, has been shown in theorem 2 and theorem 3. For the further development of this research, it is expected to show that any $a, b,$ and c are natural numbers with $a = 2$ and $2 \le b \le c$ is a super edge-magic total of theta graph $\theta(2,b,c)$ if $b, c \equiv 2\ (mod\ 4)$ for $b + c \equiv 0\ (mod\ 4)$ and $b, c \equiv 3\ (mod\ 4)$ for $b + c \equiv 2\ (mod\ 4)$.

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